DEHN FUNCTIONS OF HIGHER RANK ARITHMETIC GROUPS OF TYPE A_N IN PRODUCTS OF SIMPLE LIE GROUPS

by

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ABSTRACT

Suppose Γ is an arithmetic group defined over a global field K, that the K-type of Γ is A_n with $n \geq 2$, and that the ambient semisimple group that contains Γ as a lattice has at least two noncocompact factors. We use results from Bestvina-Eskin-Wortman and Cornulier-Tessera to show that Γ has a polynomially bounded Dehn function.

For my parents, Laura and Ian.

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CHAPTER 1

INTRODUCTION

Let K be a global field, and S a finite, nonempty set of inequivalent valuations on K. Denote by $\mathcal{O}_S$ the ring of S-integers in K, and let K_v be the completion of K with respect to $v \in S$. Let $\mathbf{G}$ be a noncommutative absolutely almost simple K-isotropic K-group, and let $G = \prod_{v \in S} \mathbf{G}(K_v)$. Note that $|S|$ is the number of simple factors of G, and that $\mathbf{G}(\mathcal{O}_S)$ is a lattice in G under the diagonal embedding.

If L is a field, the *L-rank* of $\mathbf{G}$, denoted $\mathrm{rank}_L(\mathbf{G})$ is the dimension of a maximal torus in $\mathbf{G}(L)$. The *geometric rank* of G is $k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v}(\mathbf{G})$. The Lie group G is endowed with a left invariant metric, which we will denote d_G. Lubotzky-Mozes-Raghunathan showed that if $k(\mathbf{G}, S) \geq 2$, then the word metric on $\mathbf{G}(\mathcal{O}_S)$ is Lipschitz equivalent to the restriction of d_G to $\mathbf{G}(\mathcal{O}_S)$ [LMR00].

The following is a slight generalization of a conjecture due to Gromov [Gro93]:

Conjecture 1. *If $k(\mathbf{G}, S) \geq 3$, then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is quadratic.*

Druţu showed that if $k(\mathbf{G}, S) \geq 3$, $\mathrm{rank}_K(\mathbf{G}) = 1$, and S contains only archimedean valuations, then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is bounded above by the function $x \mapsto x^{2+\epsilon}$ for any $\epsilon > 0$ [Dru98].

Young showed that if $\mathbf{G}(\mathcal{O}_S)$ is $\mathbf{SL_n}(\mathbb{Z})$ and $n \geq 5$ (i.e. $k(\mathbf{G}, S) \geq 4$), then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is quadratic [You13]. Cohen showed that if $\mathbf{G}(\mathcal{O}_S)$ is $\mathbf{Sp_{2n}}(\mathbb{Z})$ and $n \geq 5$ (i.e. $k(\mathbf{G}, S) \geq 5$), then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is quadratic [Coh14]. Bestvina-Eskin-Wortman showed that if $|S| \geq 3$ (that is, G has at least 3 factors, which implies that $k(\mathbf{G}, S) \geq 3$)), then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is polynomially bounded [BEW13].

In this paper, we will show:

Theorem 2. *If the K-type of $\mathbf{G}$ is A_n, $n \geq 2$, and $|S| \geq 2$, then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is bounded by a polynomial of degree $3 \cdot 2^n$.*

(Note that n is the K-rank of $\mathbf{G}$, and therefore $k(\mathbf{G}, S) \geq 4$.)

For example, Theorem 2 implies that the following groups have polynomially bounded Dehn functions: $\mathbf{SL_3}(\mathbb{Z}[\sqrt{2}])$, or more generally $\mathbf{SL_{n+1}}(\mathcal{O}_K)$ where $n \geq 2$, $\mathcal{O}_K$ is a ring of algebraic integers in a number field K, and $\mathcal{O}_K$ is not isomorphic to $\mathbb{Z}$ or $\mathbb{Z}[i]$; $\mathbf{SL_{n+1}}(\mathbb{Z}[1/k])$ where $n \geq 2$ and $k \in \mathbb{N} - \{1\}$; and $\mathbf{SL_{n+1}}(\mathbb{F}_p[t, t^{-1}])$ where $n \geq 2$ and p is prime. Indeed, $\mathbf{SL_{n+1}}$ is of type A_n regardless of the relative global field K, and $\mathbb{Z}[\sqrt{2}]$, $\mathcal{O}_K$, $\mathbb{Z}[1/k]$, and $\mathbb{F}_p[t, t^{-1}]$ are rings of S-integers with $|S| \geq 2$.

1.1 Dehn Functions and Isoperimetric Inequalities

If H is a finitely presented group, and w is a word in the generators of H which represents the identity, then there is a finite sequence of relators which reduces w to the trivial word. Let $\delta_H(w)$ be the minimum number of steps to reduce w to the trivial word. The *Dehn function* of H is defined as

$$\delta_H(n) = \max_{length(w) \leq n} \delta_H(w)$$

While the Dehn function depends on the presentation of H, the growth class of the Dehn function is a quasi-isometry invariant of H.

The Dehn function of a simply connected CW-complex X is defined analogously. For any loop $\gamma \subset X$, let $\delta_X(\gamma)$ be the minimal area of a disk in X that fills γ. The Dehn function of X is then

$$\delta_X(n) = \max_{length(\gamma) \leq n} \delta_X(\gamma)$$

If X is quasi-isometric to H (for example, if X has a free, cellular, properly discontinuous, cocompact H-action), then the growth class of $\delta_X(n)$ is the same as that of $\delta_H(n)$.

1.2 Coarse Manifolds

An r-*coarse manifold* in a metric space X is the image of a map from the vertices of a triangulated manifold M into X, with the property that any pair of adjacent vertices in M are mapped to within distance r of each other. We will abuse notation slightly and refer to the image of the map as an r-coarse manifold as well. If Σ is a coarse manifold, then $\partial\Sigma$ is the restriction of the map to ∂M. If M is an n-manifold, we will say Σ is a coarse n-manifold, and we define the length or area of Σ to be the number of vertices in Σ. We say two coarse n-manifolds, Σ and Σ', have the same topological type if the underlying manifolds M and M' have the same topological type.

1.3 Bounds

We will write $a = O(C)$ to mean that there is some constant k, which depends only on G and $\mathbf{G}(\mathcal{O}_S)$, such that $a \leq kC$.

CHAPTER 2

PRELIMINARIES

In this chapter, we will introduce the main tools in the proof of Theorem 2.

2.1 Parabolic Subgroups

Let K, S, and $\mathbf{G}$ be as above. There is a minimal K-parabolic subgroup $\mathbf{P} \leqslant \mathbf{G}$, and $\mathbf{P}$ contains a maximal K-split torus which we will call $\mathbf{A}$. Let Φ be the root system for $(\mathbf{G}, \mathbf{A})$, and observe that $\mathbf{P}$ determines a positive subset $\Phi^+ \subset \Phi$. Let Δ denote the set of simple roots in Φ^+. (Note that $|\Delta| = rank_K(\mathbf{G}) = n$.) For any $I \subseteq \Delta$, $[I]$ will denote the linear combinations generated by I . Let $\Phi(I)^+ = \Phi^+ - [I]$ and $[I]^+ = [I] \cap \Phi^+$. If $\alpha \in \Phi$, let $\mathbf{U}_{(\alpha)}$ be the corresponding root group. For any $\Psi \subseteq \Phi^+$ which is closed under addition, let

$$\mathbf{U}_\Psi = \prod_{\alpha \in \Psi} \mathbf{U}_{(\alpha)}$$

Note that

$$\prod_{v \in S} \mathbf{U}_\Psi(K_v)$$

can be topologically identified with a product of vector spaces and therefore can be endowed with a norm, $|| \cdot ||$.

Let $\mathbf{A}_I$ be the connected component of the identity in $(\cap_{\alpha \in I} \ker(\alpha))$. The centralizer of $\mathbf{A}_I$ in $\mathbf{G}$, $\mathbf{Z}_\mathbf{G}(\mathbf{A}_I)$, can be written as $\mathbf{Z}_\mathbf{G}(\mathbf{A}_I) = \mathbf{M}_I \mathbf{A}_I$, where $\mathbf{M}_I$ is a reductive K-group with K-anisotropic center. Notice that $\mathbf{M}_I \mathbf{A}_I$ normalizes $\mathbf{U}_{\Phi(I)^+}$, and $\mathbf{M}_I$ commutes with $\mathbf{A}_I$. We define the standard parabolic subgroup $\mathbf{P}_I$ of $\mathbf{G}$ to be

$$\mathbf{P}_I = \mathbf{U}_{\Phi(I)^+} \mathbf{M}_I \mathbf{A}_I$$

Note that $\mathbf{P}_\emptyset = \mathbf{P}$ and that when $\alpha \in \Delta$, $\mathbf{P}_{\Delta - \alpha}$ is a maximal proper K-parabolic subgroup of $\mathbf{G}$.

We will use unbolding to denote taking the product over S of the local points of a K-group, as in

$$G = \prod_{v \in S} \mathbf{G}(K_v)$$

2.2 Parabolic Regions and Reduction Theory

The following theorem was proved in different cases by Borel, Behr, and Harder (cf. [Bor91] Proposition 15.6, [Beh69] Satz 5 and Satz 8, and [Har69] Korrolar 2.2.7). A summary of the individual results and how they imply the theorem is given in [BEW13].

Theorem 3 (Borel, Behr, Harder)**.** *There is a finite set $F \subseteq \mathbf{G}(K)$ of coset representatives for $\mathbf{G}(\mathcal{O}_S)\backslash\mathbf{G}(K)/\mathbf{P}(K)$.*

Any proper K-parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ is conjugate to $\mathbf{P}_I$ for some $I \subsetneq \Delta$. Let

$$\Lambda_{\mathbf{Q}} = \{\gamma f \in \mathbf{G}(\mathcal{O}_S)F | (\gamma f)^{-1}\mathbf{P}_I(\gamma f) = \mathbf{Q} \text{ for some } I \subsetneq \Delta\}$$

By Theorem 3, $\Lambda_{\mathbf{Q}}$ is nonempty. For $a \in A$ and $\alpha \in \Phi$, let

$$|\alpha(a)| = \prod_{v\in S} |\alpha(a)|_v$$

where $|\cdot|_v$ is the norm on K_v. For $t > 0$ and $I \subset \Delta$, let

$$A_I^+(t) = \{a \in A_I \mid |\alpha(a)| \geq t \text{ if } \alpha \in \Delta - I\}$$

and $A_I^+ = A_I^+(1)$. We define the *parabolic region* corresponding to $\mathbf{Q}$ to be

$$R_{\mathbf{Q}}(t) = \Lambda_{\mathbf{Q}} U_{\Phi(I)^+} \mathbf{M}_I(\mathcal{O}_S) A_I^+(t)$$

The boundary of $A_I^+(t)$ is

$$\partial A_I^+(t) = \{a \in A_I^+ \mid \text{ there exists } \alpha \in \Delta - I \text{ with } |\alpha(a)| \leq |\alpha(b)| \text{ for all } b \in A_I^+\}$$

and the boundary of the parabolic region $R_{\mathbf{Q}}(t)$ is

$$\partial R_{\mathbf{Q}}(t) = \Lambda_{\mathbf{Q}} U_{\Phi(I)^+} \mathbf{M}_I(\mathcal{O}_S) \partial A_I^+(t)$$

For $0 \leq m < |\Delta|$, let $\mathcal{P}(m)$ be the set of K-parabolic subgroups of $\mathbf{G}$ that are conjugate via $\mathbf{G}(K)$ to some $\mathbf{P}_I$ with $|I| = m$. The necessary reduction theory is proved in [BEW13]:

Theorem 4 (Bestvina-Eskin-Wortman, 2013)**.** *There exists a bounded set $B_0 \subseteq G$, and given a bounded set $B_m \subseteq G$ and any $N_m \geq 0$ for $0 \leq m < |\Delta|$, there exists $t_m > 1$ and a bounded set $B_{m+1} \subseteq G$ such that*

(i) $G = \cup_{\mathbf{Q}\in\mathcal{P}(0)} R_{\mathbf{Q}} B_0$;

(ii) if $\mathbf{Q}, \mathbf{Q}' \in \mathcal{P}(m)$ and $\mathbf{Q} \neq \mathbf{Q}'$, then the distance between $R_{\mathbf{Q}}(t_m)B_n$ and $R_{\mathbf{Q}'}(t_m)B_n$ is at least N_m;

(iii) $\mathbf{G}(\mathcal{O}_S) \cap R_{\mathbf{Q}}(t_m)B_m = \emptyset$ *for all* m*;*

(iv) *if* $m \leq |\Delta| - 2$ *then* $(\cup_{\mathbf{Q}\in\mathcal{P}(m)} R_{\mathbf{Q}} B_m) - (\cup_{\mathbf{Q}\in\mathcal{P}(m)} R_{\mathbf{Q}}(2t_m)B_m)$ *is contained in* $\cup_{\mathbf{Q}\in\mathcal{P}(m+1)} R_{\mathbf{Q}} B_{m+1}$*;*

(v) $(\cup_{\mathbf{Q}\in\mathcal{P}(|\Delta|-1)} R_{\mathbf{Q}} B_{|\Delta|-1}) - (\cup_{\mathbf{Q}\in\mathcal{P}(|\Delta|-1)} R_{\mathbf{Q}}(2t_{|\Delta-1})B_{|\Delta|-1})$ *is contained in* $\mathbf{G}(\mathcal{O}_S)B_{|\Delta|}$*; and*

(vi) *if* $\mathbf{Q} \in \mathcal{P}(m)$*, then there is an* (L, C) *quasi-isometry* $R_{\mathbf{Q}}(t_m)B_m \to U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_I^+$ *for some* $I \subsetneq \Delta$ *with* $|I| = m$*. The quasi-isometry restricts to an* (L, C) *quasi-isometry* $\partial R_{\mathbf{Q}}(t_m)B_m \to U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)\partial A_I^+$ *where* $L \geq 1$ *and* $C \geq 0$ *are independent of* $\mathbf{Q}$*.*

2.3 Filling Coarse Manifolds

For $I \subsetneq \Delta$, we let $R_I = U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_I^+$.

Proposition 5. *Suppose* $I \subsetneq \Delta$ *is a set of simple roots, and let* R_I *denote the corresponding parabolic region of* $\mathbf{G}$*. Given* $r > 0$*, there is some* $r' \in \mathbb{R}^{>0}$ *such that if* $\Sigma \subset R_I$ *is an* r*-coarse 2-manifold of area* L*, then there is an* r'*-coarse 2-manifold* $\Sigma' \subset \partial R_I$ *such that* $\partial\Sigma = \partial\Sigma'$*. Furthermore, if* $|I| \leq |\Delta| - 2$*, then area*$(\Sigma') = O(L^2)$ *and if* $|I| = |\Delta| - 1$*, then area*$(\Sigma') = O(L^3)$*.*

Proposition 5 is proved in Sections 3.1 (for nonmaximal parabolics) and 3.2 (for maximal parabolics).

2.4 Proof of the Main Result

That Proposition 5 implies Theorem 2 is essentially proved in Bestvina-Eskin-Wortman (see [BEW13] Sections 6 and 7). We restate it here in the specific case we require, and add explicit bounds on filling areas.

Proof of Theorem 2. Let X be a simply connected CW-complex on with a free, cellular, properly discontinuous and cocompact $\mathbf{G}(\mathcal{O}_S)$-action. Let $x \in X$ be a basepoint, and define the orbit map

$$\phi : \mathbf{G}(\mathcal{O}_S) \to \mathbf{G}(\mathcal{O}_S) \cdot x$$

Note that ϕ is a bijective quasi-isometry between $\mathbf{G}(\mathcal{O}_S)$ with the left invariant metric d_G and the orbit $\mathbf{G}(\mathcal{O}_S) \cdot x$ with the path metric from X.

Let $\ell \subset X$ be a cellular loop. The $\mathbf{G}(\mathcal{O}_S)$-action on X is cocompact, so every point in ℓ is a uniformly bounded distance from $\mathbf{G}(\mathcal{O}_S)$. Therefore, there is a constant $r_0 > 0$

such that after a uniformly bounded perturbation, $\ell \cap \mathbf{G}(\mathcal{O}_S)$ is an r_0-coarse loop and the Hausdorff distance between ℓ and $\ell \cap \mathbf{G}(\mathcal{O}_S)$ is bounded. Let L be the length of $\ell \cap \mathbf{G}(\mathcal{O}_S)$.

There is a constant $r_1 > 0$ which depends only on r_0 and the quasi-isometry constants of ϕ such that $\phi^{-1}(\ell \cap \mathbf{G}(\mathcal{O}_S))$ is an r_1-coarse loop in $\mathbf{G}(\mathcal{O}_S)$. Since G is quasi-isometric to a $CAT(0)$ space (a product of Euclidean buildings and symmetric spaces), there is an r_1-coarse disk $D \subset G$ with $\partial D = \phi^{-1}(\ell \cap \mathbf{G}(\mathcal{O}_S) \cdot x)$ and area $O(L^2)$.

Set $D = D_0$ and $N_0 = 2r_1$. Let B_0 and t_0 be as in Theorem 4. If $\mathbf{Q} \in \mathcal{P}(0)$, let

$$D_{0,\mathbf{Q}} = D_0 \cap R_{\mathbf{Q}}(t_0)B_0$$

Note that $D_{0,\mathbf{Q}}$ and $D_{0,\mathbf{Q}'}$ are disjoint if $\mathbf{Q} \neq \mathbf{Q}'$. For each $\mathbf{Q} \in \mathcal{P}(0)$, we can perturb $D_{0,\mathbf{Q}}$ by at most r_1 to ensure that $\partial D_{0,\mathbf{Q}} \subset \partial R_{\mathbf{Q}}(t_0)B_0$. By Proposition 4(vi), $\partial R_{\mathbf{Q}}(t_0)B_0$ is quasi-isometric to $\partial R_{\emptyset}$. By Proposition 5, there is some r_2 depending only on r_1 and the quasi-isometry constants and an r_2-coarse 2-manifold $D'_{0,\mathbf{Q}} \subset \partial R_{\mathbf{Q}}(t_0)B_0$ such that the 2-manifold obtained by replacing each $D_{0,\mathbf{Q}}$ by $D'_{0,\mathbf{Q}}$,

$$D_1 = \left(D_0 - \bigcup_{\mathbf{Q}\in\mathcal{P}(0)} D_{0,\mathbf{Q}} \right) \bigcup \left(\bigcup_{\mathbf{Q}\in\mathcal{P}(0)} D'_{0,\mathbf{Q}} \right)$$

is an r_1-coarse 2-disk, and $area(D_1) = O(area(D_0)^2) = O(L^4)$.

By Proposition 4(iv),

$$D_1 \subset \left(\bigcup_{\mathbf{Q}\in\mathcal{P}(0)} R_{\mathbf{Q}}B_0 \right) - \left(\bigcup_{\mathbf{Q}\in\mathcal{P}(0)} R_{\mathbf{Q}}(2t_0)B_0 \right) \subset \bigcup_{\mathbf{Q}\in\mathcal{P}(1)} R_{\mathbf{Q}}B_1$$

By Proposition 4(iii), $\mathbf{G}(\mathcal{O}_S) \cap R_{\mathbf{Q}}(t_0)B_0 = \emptyset$, and therefore $\partial D_0 = \partial D_1$.

For $1 \leq m \leq |\Delta| - 1$ repeat the above process with m in place of 0, to obtain an r_{m+1}-coarse disk D_{m+1} with $\partial D_{m+1} = \partial D_0$ and $area(D_{m+1}) = O(L^{k_{m+1}})$, where $k_{m+1} = 2^{m+2}$ if $n \leq |\Delta| - 2$ and $k_{|\Delta|} = 3 \cdot 2^{|\Delta|}$. Furthermore,

$$D_{m+1} \subset \bigcup_{\mathbf{Q}\in\mathcal{P}(m)} R_{\mathbf{Q}}B_m - \bigcup_{\mathbf{Q}\in\mathcal{P}(m)} R_{\mathbf{Q}}(2t_m)B_m$$

which implies that $D_{|\Delta|} \subset \mathbf{G}(\mathcal{O}_S)B_{|\Delta|}$ by Proposition 4(v).

Since $\mathbf{G}(\mathcal{O}_S)B_{|\Delta|}$ is finite Hausdorff distance from $\mathbf{G}(\mathcal{O}_S)$, there is some $r' > 0$ such that there is an r'-coarse disk $D' \subset \mathbf{G}(\mathcal{O}_S)$ with $\partial D' = \phi^{-1}(\ell \cap \mathbf{G}(\mathcal{O}_S) \cdot x)$ and $area(D') = O(L^k)$, where $k = 3 \cdot 2^{|\Delta|}$.

There is some $r'' > 0$ which depends only on r' and the quasi-isometry constants of ϕ such that $\phi(D') \subset X$ is an r''-coarse disk with boundary $\ell \cap \mathbf{G}(\mathcal{O}_S) \cdot x$. First connect pairs of adjacent vertices in $\phi(D')$ by 1-cells to obtain D'', then add 2-cells whose 1-skeleton is in D'' to obtain D'''. Note that $\partial D''' = \ell$, D''' is a bounded Hausdorff distance to $\phi(D')$, and the number of cells in D''' is $O(area(D')) = O(L^k)$ where $k = 3 \cdot 2^{|\Delta|}$. Recall that $|\Delta| = \mathrm{rank}_K(\mathbf{G}) = n$, so the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is bounded by a polynomial of degree $3 \cdot 2^n$.

$\square$

2.5 Two Key Lemmas

Lemma 6. *Given $r > 0$ sufficiently large, $I \subseteq \Delta$, and $S' \subsetneq S$, there is some $a \in \mathbf{A}_I(\mathcal{O}_S)$ that strictly contracts all root subgroups of $\prod_{v \in S'} \mathbf{U}_{\Phi(I)^+}(K_v)$, such that $d_G(a, 1) \leq r$.*

Proof. Lemma 12 in Bestvina-Eskin-Wortman [BEW13] shows that the projection of $\mathbf{A}_I(\mathcal{O}_S)$ to $\prod_{v \in S'} \mathbf{A}_I(K_v)$ is a finite Hausdorff distance from $\prod_{v \in S'} \mathbf{A}_I(K_v)$. (The proof is independent of $|S|$.) This implies that there is some $a \in \mathbf{A}_I(\mathcal{O}_S)$ such that $|\alpha(a)|_v < 1$ for all $\alpha \in \Delta - I$ and $v \in S'$. Therefore, if $u \in \prod_{v \in S'} \mathbf{U}_{(\beta)}(K_v)$ for some $\beta \in \Phi(I)^+$, then $||a^{-1}ua|| < ||u||$. $\square$

We will make use of the following lemma in both the maximal and nonmaximal parabolic cases:

Lemma 7. *Let $r > 0$ be sufficiently large and $I \subset \Delta$. If $u \in U_{\Phi(I)^+}$, then there is an r-coarse path $p_u \subset U_{\Phi(I)^+}\mathbf{A}_I^+(\mathcal{O}_S)$ joining u to 1 such that length$(p_u) = O(d_G(u, 1))$.*

Proof. Let $L = d_G(u, 1)$, and notice that $||u|| \leq O(e^L)$. Letting $S = \{v_1, \ldots, v_k\}$, we can write $u = (u_1, \ldots, u_k)$, where $u_i \in \mathbf{U}_{\Phi(I)^+}(K_{v_i})$.

By the bound on $||u||$, we also have $||u_i|| \leq O(e^L)$. By Lemma 6, we can choose $a_i \in \mathbf{A}_I^+(\mathcal{O}_S)$ such that a_i strictly contracts $\mathbf{U}_{\Phi(I)^+}(K_{v_i})$ and $d_G(a_i, 1) \leq r$.

For some $T_i = O(L)$, $d_G(a_i^{-T_i} u_i a_i^{T_i}, 1) = d_G(u_i a_i^{T_i}, a_i^{T_i}) \leq 1$. Let $p_i = \{a_i^k \mid 0 \leq k \leq T_i\} \cup \{u a_i^k \mid 0 \leq k \leq T_i\}$. Note that p_i is an r-coarse path from 1 to u_i of length $O(L)$. Taking

$$p_u = p_1 \cup \left(\bigcup_{2 \leq i \leq k} (u_1, \ldots, u_{i-1}, 1, \ldots, 1) \cdot p_i \right)$$

gives the desired path from 1 to u. $\square$

CHAPTER 3

PROOF OF PROPOSITION 5

In this chapter, we will prove Proposition 5. Section 3.1 covers the case of nonmaximal parabolic subgroups and Section 3.2 covers the case of maximal parabolic subgroups.

3.1 Nonmaximal Parabolic Subgroups

In this section, we will prove Proposition 5 in the case where $|I| \leq |\Delta| - 2$.

First, we will divide ∂R_I into two pieces. Recall that

$$\partial R_I = U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)\partial A_I^+$$

$$\partial A_I^+ = \{a \in A_I^+ | \text{ there exists } \alpha \in \Delta - I \text{ with } |\alpha(a)| \leq |\alpha(b)| \text{ for all } b \in A_I^+\}$$

For $\alpha \in \Delta - I$, we define $A_{I,\alpha}^+$, $Z_{I,\alpha}^+$, $B_{I,\alpha}$, and $\hat{B}_{I,\alpha}$ as follows:

$$A_{I,\alpha}^+ = \left\{a \in A_I^+ \mid |\alpha(a)| \leq |\alpha(b)| \text{ for all } b \in A_I^+\right\}$$

$$Z_{I,\alpha}^+ = \bigcup_{\beta \in \Delta - (I \cup \alpha)} A_{I,\beta}^+$$

$$B_{I,\alpha} = U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_{I,\alpha}^+$$

$$\hat{B}_{I,\alpha} = U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)Z_{I,\alpha}^+$$

Note that $\partial A_I^+ = \cup_{\alpha \in \Delta - I} A_{I,\alpha}^+$ and that $\partial R_I = B_{I,\alpha} \cup \hat{B}_{I,\alpha}$. We also observe that $A_{I,\alpha}^+ \neq A_{I\cup\alpha}^+$, since $\mathbf{A}_I(\mathcal{O}_S) \subseteq A_{I,\alpha}^+$ for any $\alpha \in \Delta - I$, but $\mathbf{A}_I(\mathcal{O}_S) \not\subset A_{I\cup\alpha}^+$ in general.

Since A_I^+ is quasi-isometric to a Euclidean space, there is a projection to ∂A_I^+ which is distance nonincreasing. Note that $\mathbf{M}_I(\mathcal{O}_S)$ commutes with A_I^+, so there is a distance nonincreasing map $\mathbf{M}_I(\mathcal{O}_S)A_I^+ \to \mathbf{M}_I(\mathcal{O}_S)\partial A_I^+$. Let $\pi_I : R_I \to \partial R_I$ be the composition of the distance nonincreasing maps $U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_I \to \mathbf{M}_I(\mathcal{O}_S)A_I^+$ and $\mathbf{M}_I(\mathcal{O}_S)A_I^+ \to \mathbf{M}_I(\mathcal{O}_S)\partial A_I^+$.

Lemma 8. *Suppose $I \subsetneq \Delta$ is a set of simple roots such that $|I| \leq |\Delta| - 2$ and let $r > 0$ and $\alpha \in \Delta - I$ be given. If $\Sigma \subset R_I$ is an r-coarse 2-manifold with boundary and $\partial\Sigma \subset \partial R_I$, then $\Sigma = \Sigma_1 \cup \Sigma_2$ for r-coarse 2-manifolds with boundary, Σ_1 and Σ_2, such that* *(i)* $\pi_I(\partial\Sigma_1) \subset B_{I,\alpha}$ *and* $\pi_I(\partial\Sigma_2) \subset \hat{B}_{I,\alpha}$*,*

(ii) $\Sigma_1 \cap \partial\Sigma \subset B_{I,\alpha}$ *and* $\Sigma_2 \cap \partial\Sigma \subset \hat{B}_{I,\alpha}$*, and*

(iii) $\Sigma_1 \cap \Sigma_2$ *consists of finitely many* r*-coarse paths* $p_1, \ldots, p_k$*, with* $\pi_I(p_i) \subset \partial B_{I,\alpha}$ *and finitely many* r*-coarse loops* $\gamma_1, \ldots, \gamma_n$ *with* $\pi_I(\gamma_l) \subset \partial B_{I,\alpha}$*.*

Proof. By transversality, $\pi_I(\Sigma)$ intersects $\partial B_{I,\alpha}$ in an r-coarse 1-manifold which is made up of finitely many r-coarse paths $(\bar{p}_1, \ldots, \bar{p}_k)$ with endpoints in $\pi_I(\partial\Sigma)$ and finitely many r-coarse loops $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ which do not intersect $\pi_I(\partial\Sigma)$. Furthermore, $\pi_I(\Sigma)$ intersects $B_{I,\alpha}$ (respectively $\hat{B}_{I,\alpha}$) in a 2-manifold with boundary, $\bar{\Sigma}_1$ (respectively $\bar{\Sigma}_2$), and

$$\partial\bar{\Sigma}_i = (\bar{\Sigma}_i \cap \pi_I(\partial\Sigma)) \cup (\bar{p}_1 \cup \cdots \cup \bar{p}_k) \cup (\bar{\gamma}_1 \cup \cdots \cup \bar{\gamma}_n) \tag{3.1}$$

For $x \in \partial R_I$, note that $\pi_I(x) \in B_{I,\alpha}$ if and only if $x \in B_{I,\alpha}$ (since π_I only changes the unipotent coordinates of points in ∂R_I). Let Σ_1 and Σ_2 be the respective preimages of $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ under π_I restricted to Σ. Note that $\bar{p}_i$ and $\bar{\gamma}_i$ lift to r-coarse paths and loops p_i and γ_i in Σ. Conclusion (i) holds because $\bar{\Sigma}_i = \pi_I(\Sigma_i)$, and conclusions (ii) and (iii) hold by (1) and the definition of p_i and γ_l. $\square$

Lemma 9. *Suppose* $I \subsetneq \Delta$ *is a set of simple roots such that* $|I| \leq |\Delta| - 2$ *and let* $r > 0$ *and* $\alpha \in \Delta - I$ *be given. If* Ω *is a closed* r*-coarse* 1*-manifold in* $B_{I,\alpha}$ *or* $\hat{B}_{I,\alpha}$ *with diameter and distance to* $B_{I,\alpha} \cap \hat{B}_{I,\alpha}$ *bounded by* L*, then there is an* r'*-coarse* 2*-manifold* $\mathcal{A} \subset \partial R_I$ *such that* $\partial\mathcal{A} = \Omega \cup u\pi_I(\Omega)$ *for some* $u \in U_{\Phi(I)^+}$ *and area*$(\mathcal{A}) = O(L^2)$*.*

Proof. We will begin with the case where $\Omega \subset B_{I,\alpha}$. For $x \in \Omega$, we can write $x = u_x m_x a_x$ with $u_x \in U_{\Phi(I)^+}, m_x \in \mathbf{M}_I(\mathcal{O}_S)$, and $a_x \in A^+_{I,\alpha}$. Since the diameter of Ω is bounded by L, $||u_x^{-1}u_y|| \leq O(e^L)$ for any $x, y \in \Omega$. Choose $b \in int(A^+_{I\cup\alpha})$ with $d_G(b,1) \leq r$. Note that b commutes with $U_{[I\cup\alpha]}$, $\mathbf{M}_I(\mathcal{O}_S)$, and A^+_I, and that conjugation by b^{-1} strictly contracts $U_{\Phi(I\cup\alpha)^+}$. Also, $U_{\Phi(I)^+} = U_{\Phi(I\cup\alpha)^+}U_{[I\cup\alpha]\cap\Phi(I)^+}$, so conjugation by b^{-1} does not expand any root group in $U_{\Phi(I)^+}$.

Since $d_G(b,1) \leq r$, left invariance of d_G implies that $d_G(gb, g) \leq r$ for any $g \in G$. Right multiplication by b^k is distance nonincreasing on Ω when $k \geq 0$, since for any $x, y \in \Omega$,

$$\begin{aligned} d_G(xb^k, yb^k) &= d_G(u_x m_x a_x b^k, u_y m_y a_y b^k) \\ &= d_G(u_x b^k m_x a_x, u_y b^k m_y a_y) \\ &= d_G(b^{-k} u_y^{-1} u_x b^k m_x a_x, m_y a_y) \\ &\leq d_G(u_y^{-1} u_x m_x a_x, m_y a_y) \\ &= d_G(u_x m_x a_x, u_y m_y a_y) \\ &= d_G(x, y) \end{aligned}$$

Therefore, $\cup_{0 \leq k \leq m} \Omega b^k$ is a $2r$-coarse 2-manifold for any $m \in \mathbb{N}$, which has the topological type of $\Omega \times [0,1]$, boundary $\Omega \cup \Omega b^n$, and whose area is bounded by Lm. There is some $T = O(L)$ such that the $U_{\Phi(I \cup \alpha)^+}$-coordinates of Ωb^T are nearly constant. More precisely, there is some fixed $u^* \in U_{\Phi(I\cup\alpha)^+}$ and some $v_x \in U_{[I\cup\alpha]\cap\Phi(I)^+}$ for each x such that

$$d_G(u_x m_x a_x b^T, u^* v_x m_x a_x b^T) \leq r$$

for every x in Ω. Let $\Omega_1 = \{u^* v_x m_x a_x b^T\}_{x \in \Omega}$ and let $\mathcal{A}_1 = \Omega_1 \cup \; (\cup_{0 \leq k \leq T} \Omega b^k)$ be the $2r$-coarse 2-manifold with boundary $\Omega \cup \Omega_1$. Note that $area(\mathcal{A}_1) = O(L^2)$.

Let $\Omega_2 = \{u^* v_x m_x a_x\}_{x \in \Omega}$. Note that Ω_2 is an r-coarse 1-manifold of the same diameter as Ω, since

$$\begin{aligned} d_G(u^* v_x m_x a_x, u^* v_y m_y a_y) &= d_G(u^* b^T v_x m_x a_x, u^* b^T v_y m_y a_y) \\ &= d_G(u^* v_x m_x a_x b^T, u^* v_y m_y a_y b^T) \\ &\leq r \end{aligned}$$

Again, there is a $2r$-coarse 2-manifold formed by $\cup_{0 \leq k \leq T} \Omega_1 b^{-k}$, with area $O(L^2)$ and boundary $\Omega_1 \cup \Omega_2$. After left translation, $(u^*)^{-1}\Omega_2 \subset U_{[I\cup\alpha]\cap\Phi(I)^+} \mathbf{M}_I(\mathcal{O}_S) A^+_{I,\alpha}$. Since b commutes with $U_{[I\cup\alpha]\cap\Phi(I)^+} \mathbf{M}_I(\mathcal{O}_S) A^+_{I,\alpha}$, after a perturbation by at most r, the $2r$-coarse 2-manifold formed by $\cup_{k \in \mathbb{Z}} (u^*)^{-1} \Omega_2 b^k$ intersects $\partial B_{I,\alpha}$ in a $2r$-coarse closed 1-manifold of length $O(L)$. Call this Ω_3 and let $\mathcal{A}_3$ be the portion of $\cup_{k\in\mathbb{Z}} (u^*)^{-1}\Omega_2 b^k$ bounded by $(u^*)^{-1}\Omega_2$ and Ω_3. Since the distance from Ω to $\partial B_{I,\alpha}$ is bounded by L, the area of $\mathcal{A}_3$ is $O(L^2)$. Note that if $\hat{x} = v_x m_x a_x \in \; (u^*)^{-1}\Omega_2$, then $\bar{x} = v_x m_x \bar{a}_x \in \Omega_3$, where $\bar{a}_x \in \partial A^+_{I,\alpha}$. The bound on the diameter of Ω_3 implies that $||v_x^{-1} v_y|| \leq e^L$ for all $\bar{x} \in \Omega_3$.

Choose $c \in \partial A^+_I$ such that $d_G(c, 1) \leq r$, and for every $v \in S$, $|\alpha(c)|_v > 1$ and $|\beta(c)|_v = 1$ for every $\beta \in \Delta - \alpha$. There is some $T' = O(L)$ such that $\Omega_3 c^{T'}$ has nearly constant $U_{[I\cup\alpha]\cap\Phi(I)^+}$-coordinates. That is, there is some $v^* \in U_{[I\cup\alpha]\cap\Phi(I)^+}$ such that $d_G(v_x m_x \bar{a}_x c^{T'}, v^* m_x \bar{a}_x c^{T'}) \leq 2r$ for all $\bar{x} \in \Omega_3$. Let $\Omega_4 = \{v^* m_x \bar{a}_x c^{T'}\}_{x \in \Omega}$, and let $\mathcal{A}_4$ be the $4r$-coarse 2-manifold $\Omega_4 \cup (\cup_{0 \leq k \leq T'} \Omega_3 c^k)$. The area of $\mathcal{A}_4$ is $O(L^2)$. Since c commutes with $\mathbf{M}_I(\mathcal{O}_S)$ and A_I^+, $\Omega_5 = \Omega_4 c^{-T'}$ is a $2r$-coarse 1-manifold, and there is a $4r$-coarse 2-manifold $\mathcal{A}_5 = \cup_{0\leq k\leq T'} \Omega_4 c^{-k}$ which has boundary $\Omega_4 \cup \Omega_5$, and area $O(L^2)$.

Finally, observe that $\Omega_5 = \{v^* m_x \bar{a}_x\}_{x\in\Omega}$ has the same $\mathbf{M}_I(\mathcal{O}_S)$-coordinates as Ω, and that b commutes with Ω_5. Therefore, there is a $2r$-coarse 1-manifold $\Omega_6 \subset \cup_{k\in\mathbb{Z}} \Omega_5 b^k$ which has the form $\Omega_6 = \{v^* m_x a_x\}_{x\in\Omega}$, and there is a $4r$-coarse 2-manifold $\mathcal{A}_6$ bounded by Ω_5 and Ω_6 with area $O(L^2)$.

Taking

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup u^*\mathcal{A}_3 \cup u^*\mathcal{A}_4 \cup u^*\mathcal{A}_5 \cup u^*\mathcal{A}_6$$

and $r' = 4r$ completes the proof. □

Lemma 10. *Suppose $I \subsetneq \Delta$ is a set of simple roots such that $|I| \leq |\Delta| - 2$, and let $\alpha \in \Delta - I$ and $r > 0$ be given. If $p \subset R_I$ is an r-coarse path with endpoints $x, y \in \partial B_{I,\alpha}$ such that $\pi_I(p) \subset \partial B_{I,\alpha}$, then there is an r-coarse path $p' \subset \partial B_{I,\alpha}$ joining x to y of length $O(length(p))$, and $\pi_I(p) \cup \pi_I(p')$ bound a disk of area $O(length(p)^2)$ in ∂R_I.*

Proof. Let $length(p) = L$. We can write $x = u_x m_x a_x$ and $y = u_y m_y a_y$ for $u_x, u_y \in U_{\Phi(I)^+}$; $m_x, m_y \in \mathbf{M}_I(\mathcal{O}_S)$; and $a_x, a_y \in \partial A^+_{I,\alpha}$.

Since π_I is distance nonincreasing, $\pi_I(p)$ is an r-coarse path of length L from $m_x a_x$ to $m_y a_y$. Left multiplication by u_x gives an r-coarse path p_1, with length L, joining x to $u_x m_y a_y$.

Note that $u' = (m_y a_y)^{-1}(u_x^{-1} u_y)(m_y a_y) \in U_{\Phi(I)^+}$ because $U_{\Phi(I)^+}$ is normalized by $\mathbf{M}_I(\mathcal{O}_S) A_I^+$. Also,

$$\begin{aligned} d_G(u', 1) &= d_G(u_x m_y a_y, u_y m_y a_y) \\ &\leq d_G(u_x m_y a_y, u_x m_x a_x) + d_G(u_x m_x a_x, u_y m_y a_y) \\ &\leq 2L \end{aligned}$$

By Lemma 7, there is an r-coarse path in $U_{\Phi(I)^+}\mathbf{A}_I^+(\mathcal{O}_S)$ from u' to 1, with length $O(L)$. Left multiplication by $u_x m_y a_y$ gives an r-coarse path $p_2 \subset m_y a_y U_{\Phi(I)^+}\mathbf{A}_I^+(\mathcal{O}_S)$ of length $O(L)$ joining $u_x m_y a_y$ to y.

Let $p' = p_1 \cup p_2$. Note that $p \cup p'$ is a loop in R_I, and that $\pi_I(p_1) = \pi_I(p)$. Therefore $\pi_I(p_2)$ forms a loop in $m_y \mathbf{A}_I^+(\mathcal{O}_S)$. Since $\mathbf{A}_I^+(\mathcal{O}_S)$ is quasi-isometric to a Euclidean space of dimension $(|\Delta - I|)(|S| - 1)$, it has a quadratic Dehn function, and therefore $\pi_I(p_2)$ bounds an r-coarse disk of area $O(L^2)$ in $m_y \mathbf{A}_I^+(\mathcal{O}_S) \subset \partial R_I$. □

We will now prove Proposition 5 in the case when $|I| \leq |\Delta| - 2$.

Proof of Proposition 5 for nonmaximal parabolics. We will prove the lemma in two cases: first the case where $\partial\Sigma$ intersects both $B_{I,\alpha}$ and $\hat{B}_{I,\alpha}$ nontrivially for some $\alpha \in \Delta - I$; second the case where $\partial\Sigma \subset B_{I,\alpha}$ for some $\alpha \in \Delta - I$. These two cases are sufficient, because $\partial R_I = \cup_{\alpha\in\Delta-I} B_{I,\alpha}$, so $\partial\Sigma$ must intersect $B_{I,\alpha}$ for at least one $\alpha \in \Delta - I$.

Suppose Σ intersects both $B_{I,\alpha}$ and $\hat{B}_{I,\alpha}$. By Lemma 8, Σ can be written as the union of two r-coarse 2-manifolds, Σ_1 and Σ_2, such that $\Sigma_1 \cap \partial\Sigma \subset B_{I,\alpha}$ and $\Sigma_2 \cap \partial\Sigma \subset \hat{B}_{I,\alpha}$, and $\Sigma_1 \cap \Sigma_2$ is a collection of r-coarse loops and r-coarse paths in R_I with endpoints in $\partial\Sigma$.

Suppose p_j is an r-coarse path in $\Sigma_1 \cap \Sigma_2$, with endpoints in $\partial B_{I,\alpha}$. Lemma 8 implies that $\pi_I(p_j) \subset \partial B_{I,\alpha}$, so we can apply Lemma 10 to obtain an r-coarse path p'_j in $\partial B_{I,\alpha}$ which has the same endpoints as p_j and length $O(length(p_j))$. If γ_l is an r-coarse loop in $\Sigma_1 \cap \Sigma_2$, choose $x_l \in \gamma_l$ and write $x_l = u_l g_l$ for $u_l \in U_{\Phi(I)^+}$ and $g_l \in \mathbf{M}_I(\mathcal{O}_S)A_I^+$. Let $\gamma'_l = u_l \pi_I(\gamma_l)$ and note that $\gamma'_l \subset \partial B_{I,\alpha}$ and $\pi_I(\gamma'_l) = \pi_I(\gamma_l)$.

Note that $\partial\Sigma_i$ is a closed 1-manifold, and

$$\partial\Sigma_i = (\Sigma_i \cap \partial\Sigma) \cup (p_1 \cup \cdots \cup p_k) \cup (\gamma_1 \cup \cdots \cup \gamma_n)$$

Although $\partial\Sigma_i \not\subset \partial R_I$, we can replace p_j by p'_j and γ_l by γ'_l to obtain a closed 1-manifold of the same topological type as $\partial\Sigma_i$ which is contained in ∂R_I. Let

$$\Omega_i = (\Sigma_i \cap \partial\Sigma) \cup (p'_1 \cup \cdots \cup p'_k) \cup (\gamma'_1 \cup \cdots \cup \gamma'_n)$$

By Lemmas 8 and 10, the total length of Ω_i is $O(area(\Sigma))$.

Lemma 9 implies the existence of a constant $r' > 0$ and r'-coarse 2-manifolds $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\partial\mathcal{A}_i = \Omega_i \cup u_i\pi_I(\Omega_i)$ for some $u_i \subset U_{\Phi(I)^+}$, and $area(\mathcal{A}_i) = O(area(\Sigma)^2)$. By Lemma 10, there is a family of disks $D_{i,j} \subset \partial R_I$ such that

$$\Sigma'_i = \mathcal{A}_i \cup (\cup_j D_{i,j}) \cup u_i\pi_I(\Sigma_i)$$

is an r'-coarse 2-manifold of the same topological type as Σ_i. Note that $\sum_{j=1}^k length(p_j) \leq L$, which implies that $\sum_{j=1}^k area(D_{i,j}) \leq L^2$ and therefore $area(\Sigma'_i) = O(area(\Sigma)^2)$. Taking $\Sigma' = \Sigma'_1 \cup \Sigma'_2$ completes the first case of the proof.

We now assume that $\partial\Sigma \subset B_{I,\alpha}$. Let $\Omega = \partial\Sigma$ and let L be the total length of $\partial\Sigma$. Every point $x \in \partial\Sigma$ can be written as $x = u_x m_x a_x$ for $u_x \in U_{\Phi(I)^+}$, $m_x \in \mathbf{M}_I(\mathcal{O}_S)$, and $a_x \in A^+_{I,\alpha}$. Note that $||u_x^{-1}u_y|| = O(e^L)$ for $x, y \in \partial\Sigma$. Choose some $b \in int(A^+_{I\cup\alpha})$ which strictly contracts $U_{\Phi(I\cup\alpha)^+}$. As in the proof of Lemma 9, right multiplication by b^k is distance nonincreasing on Σ when $k \geq 0$, and there is some $T = O(L)$ such that Ωb^T has nearly constant $U_{\Phi(I \cup \alpha)^+}$-coordinates. Let $u^* \in U_{\Phi(I \cup \alpha)^+}$ be such that

$$d_G(u_x m_x a_x b^T, u^* v_x m_x a_x b^T) \leq r$$

for every $x \in \Omega$. Let $\Omega_1 = \{u^* v_x m_x a_x | x \in \Omega\}$. As in the proof of Lemma 9, there is a $2r$-coarse 2-manifold $\mathcal{A}$ with boundary $\Omega \cup \Omega_1$ and area $O(L^2)$.

There is a distance nonincreasing map $f : U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_I^+ \to U_{[I \cup \alpha] \cap \Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_{I,\alpha}^+$. Taking $r' = 2r$ and $\Sigma' = f(\Sigma) \cup \mathcal{A}$ completes the proof. □

3.2 Maximal Parabolic Subgroups

In this section, we will prove Proposition 5 in the case where R_I is a maximal parabolic subgroup of G (when $|I| = |\Delta| - 1$). There is a simple root $\alpha \in \Delta$ such that $I = \Delta - \alpha$.

As in the previous section, there is a distance nonincreasing map $\pi_I : U_{\Phi(I)^+}\mathbf{M}_I(\mathcal{O}_S)A_I \to \mathbf{M}_I(\mathcal{O}_S)\partial A_I$. Note that $\partial A_I = A_\Delta$ which is quasi-isometric to $\mathbf{A}(\mathcal{O}_S)$, so $\mathbf{M}_I(\mathcal{O}_S)\partial A_I$ is quasi-isometric to $(\mathbf{M}_I\mathbf{A})(\mathcal{O}_S)$.

Lemma 11. *Given $r > 0$ sufficiently large, and $x \in \partial R_I$, with $d_G(x, 1)$ bounded by L, there is an r-coarse path in ∂R_I joining x to $\pi_I(x)$ which has length $O(L)$.*

Proof. We can write $x = uma$ for $u \in U_{\Phi(I)^+}$, $m \in \mathbf{M}_I(\mathcal{O}_S)$ and $a \in \mathbf{A}(\mathcal{O}_S)$. Then $\pi_I(x) = ma$. Note that $(\mathbf{M}_I\mathbf{A})(\mathcal{O}_S)$ normalizes $U_{\Phi(I)^+}$. So finding an r-coarse path from x to $\pi_I(x)$ of length $O(L)$ can be reduced to the problem of finding an r-coarse path from $(ma)^{-1}u(ma) \in U_{\Phi(I)^+}$ to 1 of length $O(L)$. Since $||(ma)^{-1}u(ma)|| \leq O(L)$, Lemma 7 completes the proof. □

Fix some $w \in S$. Let $\mathbf{T}_I$ be a K-defined K-anisotropic torus in $\mathbf{M}_I$ such that $g\mathbf{T}_I g^{-1} = \mathbf{M}_I \cap \mathbf{A}$. Since $\mathbf{T}_I$ is K-anisotropic, Dirichlet's units theorem tells us that $\mathbf{T}_I(\mathcal{O}_S)$ is cocompact in T_I, so in particular, the projection of $\mathbf{T}_I(\mathcal{O}_S)$ to $\mathbf{T}_I(K_w)$ is a finite Hausdorff distance from $\mathbf{T}_I(K_w)$. Let $\widehat{T}_I$ be the projection of $\mathbf{T}_I(\mathcal{O}_S)$ to $\mathbf{T}_I(K_w)$.

Lemma 12. *Suppose $\beta \in \Phi(I)^+$, so that $\mathbf{U}_{(\beta)}(K_w) \leqslant \mathbf{U}_{\Phi(I)^+}(K_w)$. There is some $t \in \widehat{T}_I$ such that gtg^{-1} strictly contracts $\mathbf{U}_{(\beta)}(K_w)$.*

Proof. It suffices to show that there is some $t' \in \mathbf{M}_I(K_w) \cap \mathbf{A}(K_w)$ which strictly contracts $\mathbf{U}_{(\beta)}(K_w)$.

We first note that since the K-type of $\mathbf{G}$ is A_n, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and a general root $\gamma \in \Phi$ has the form

$$\gamma = \pm \sum_{i=j}^{k} \alpha_i$$

where $1 \leq j \leq k \leq n$. Because P_I is a maximal parabolic, $I = \Delta - \alpha_m$ for some m such that $1 \leq m \leq n$.

Let $\Delta_1 = \{\alpha_1, \ldots, \alpha_{m-1}\}$ and $\Delta_2 = \{\alpha_{m+1}, \ldots, \alpha_n\}$. At least one of these sets must be nonempty. We will assume that Δ_2 is nonempty for the sake of simplicity. We can write $\mathbf{M}_I = \mathbf{M}_1 \times \mathbf{M}_2$, where

$$\mathbf{M}_1 = \langle \mathbf{U}_{(\alpha_i)}, \mathbf{U}_{(-\alpha_i)} \rangle_{i<m}$$

$$\mathbf{M}_2 = \langle \mathbf{U}_{(\alpha_i)}, \mathbf{U}_{(-\alpha_i)} \rangle_{i>m}$$

Let $\mathbf{A}_i = \mathbf{A} \cap \mathbf{M}_i$, and note that $\mathbf{P}_\emptyset \cap \mathbf{M}_i$ is a minimal parabolic subgroup of $\mathbf{M}_i$, $\mathbf{A}_i$ is a maximal K-split torus in $\mathbf{P}_\emptyset \cap \mathbf{M}_i$, and Δ_i is the set of simple roots with respect to $\mathbf{A}_i$.

Since $\beta \in \Phi(\Delta - \alpha_m)^+$, we know that

$$\beta = \alpha_j + \cdots + \alpha_m + \cdots + \alpha_k$$

for fixed choices of j and k such that $1 \leq j \leq m \leq k \leq n$.

Suppose that $k > m$, and choose $a \in \mathbf{A}_2^+(K_w)$ such that $|\alpha_i(a)|_w < 1$ for all $\alpha_i \in \Delta_2$. Note that $|\alpha_i(a)|_w = 1$ for $\alpha_i \in \Delta_1$, since $a \in \mathbf{M}_2(K_w)$.

Conjugation by a acts on $\mathbf{U}_{(\beta)}(K_w)$ by scalar multiplication by the constant

$$C = \prod_{i=j}^{k} |\alpha_i(a)|_w$$

By our choice of a, we know that $C = |\alpha_m(a)|_w C'$ where $C' < 1$. If $|\alpha_m(a)|_w < \frac{1}{C'}$, then $C < 1$, and a contracts $\mathbf{U}_{(\beta)}(K_w)$ by a factor of C. If $|\alpha_m(a)|_w > \frac{1}{C'}$, then $C > 1$ and a^{-1} contracts $\mathbf{U}_{(\beta)}(K_w)$ by a factor of $\frac{1}{C}$. (Note that either a or a^{-1} must contract $U_{(\gamma)}(K_w)$ for any other $\gamma \in \Phi(I)^+$ with $k > m$.)

If $C = 1$, choose $a' \in \cap_{i=1}^{m} \ker(\alpha_i)$ such that $|\alpha_i(a')|_w \leq 1$ for all $\alpha_i \in \Delta_2$ and $|\alpha_k(a')|_w < 1$. Note that

$$\prod_{i=j}^{k} |\alpha_i(aa')|_w = C \prod_{i=m+1}^{k} |\alpha_i(a')|_w < C$$

so aa' contracts $\mathbf{U}_{(\beta)}(K_w)$.

If $\beta = \alpha_j + \cdots + \alpha_m$, a different approach is required. Consider the group

$$\mathbf{M}_3 = \langle \mathbf{U}_{\alpha_m}, \mathbf{U}_{-\alpha_m}, \mathbf{U}_{\alpha_{m+1}}, \mathbf{U}_{-\alpha_{m+1}} \rangle$$

and let $\mathbf{A}_3 = \mathbf{M}_3 \cap \mathbf{A}$. Note that $\Delta_3 = \{\alpha_m, \alpha_{m+1}\}$ is the set of simple roots of $\mathbf{M}_3$, and the K-type of $\mathbf{M}_3$ is A_2. Furthermore, α_m determines a maximal parabolic subgroup $\mathbf{P}^* \leqslant \mathbf{M}_3$, with $\ker(\alpha_m) = \mathbf{P}^* \cap \mathbf{A}_3$.

Let $L = \langle \mathbf{U}_{\alpha_{m+1}}(K_w), \mathbf{U}_{-\alpha_{m+1}}(K_w) \rangle$, and choose $a \in L \cap \mathbf{A}_3(K_w)$ with $|\alpha_{m+1}(a)|_w < 1$. We argue that a contracts $\mathbf{U}_{(\beta)}(K_w)$. Since $L \cap \mathbf{A}_1(K_w)$ is trivial, $|\alpha_i(a)|_w = 1$ for all $i < m$. So the action of a on $\mathbf{U}_{(\beta)}(K_w)$ depends only on $|\alpha_m(a)|_w$. Let ϕ be the K-automorphism of $\mathbf{M}_3$ which stabilizes $\mathbf{A}_3$ and transposes $\mathbf{P}^*$ with its opposite with respect to $\mathbf{A}_3$. Note that $\ker(\alpha_m) \cap L$ is trivial, since ϕ preserves L but does not preserve $\mathbf{P}^*$. Therefore, $|\alpha_m(a)|_w \neq 1$, and after possibly replacing a by its inverse, we find that a contracts $\mathbf{U}_{(\beta)}(K_w)$ by a factor of $|\alpha_m(a)|_w$.

□

Lemma 13. *The Dehn function of $U_{\Phi(I)^+}\widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$ is quadratic.*

Proof. We observe that $\widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$ is a free abelian group. Also, $U_{\Phi(I)^+}$ is normalized by $\widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$, and since the K-type of $\mathbf{G}$ is A_n, $U_{\Phi(I)^+}$ is abelian and $\mathbf{U}_{\Phi(I)^+}(K_v)$ isomorphic to a direct sum of one or more copies of K_v.

Therefore, $U_{\Phi(I)^+}\widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$ can be written as

$$\bigoplus_{v \in S} \mathbf{U}_{\Phi(I)^+}(K_v) \rtimes \widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$$

By Theorem 3.1 in [CT10], it suffices to show that for any two unipotent coordinate subgroups, $\mathbf{U}_{(\beta_1)}(K_v)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$, of $U_{\Phi(I)^+}$, there is some element of $\widehat{T} \mathbf{A}_I(\mathcal{O}_S)$ which simultaneously contracts $\mathbf{U}_{(\beta_1)}(K_v)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$.

If $v = v'$, then $\mathbf{U}_{(\beta_1)}(K_v)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$ are contained in the same factor of $U_{\Phi(I)^+}$. By Lemma 6, there is some $a \in \mathbf{A}_I(\mathbf{O}_S)$ which simultaneously contracts $\mathbf{U}_{(\beta_1)}(K_v)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$.

If $v \neq v'$, then $\mathbf{U}_{(\beta_1)}(K_v)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$ are in different factors of $U_{\Phi(I)^+}$. In this case, either $|S| \geq 3$ or $|S| = 2$. If $|S| \geq 3$, then we may again apply Lemma 6 to obtain $a \in \mathbf{A}_I(\mathcal{O}_S)$ which simultaneously contracts $\mathbf{U}_{\Phi(I)^+}(K_v) \times \mathbf{U}_{\Phi(I)^+}(K_{v'})$.

If $|S| = 2$, we may assume that $v = w$. Let $g \in \mathbf{M}_I(K_w) \times \{1\}$ be the element which diagonalizes $\widehat{T}_I$. Note that g commutes with $\mathbf{A}_I(\mathcal{O}_S)$ and normalizes $U_{\Phi(I)^+}$, so $U_{\Phi(I)^+}\widehat{T}_I \mathbf{A}_I(\mathcal{O}_S)$ is conjugate to $U_{\Phi(I)^+}(g\widehat{T}_I g^{-1})\mathbf{A}_I(\mathcal{O}_S)$, and it suffices to prove the lemma for the latter group.

By Lemma 12, there is some $gtg^{-1} \in g\widehat{T}_I g^{-1}$ which contracts $\mathbf{U}_{(\beta_1)}(K_w)$ and commutes with $\mathbf{U}_{(\beta_2)}(K_{v'})$. There is some $a \in \mathbf{A}_I(\mathcal{O}_S)$ which contracts $\mathbf{U}_{(\beta_2)}(K_{v'})$. If a expands $\mathbf{U}_{(\beta_1)}(K_w)$, then there is a positive power of gtg^{-1} such that $gt^k g^{-1} a$ simultaneously contracts $\mathbf{U}_{(\beta_1)}(K_w)$ and $\mathbf{U}_{(\beta_2)}(K_{v'})$. $\square$

Proof of Proposition 5 for maximal parabolics. Since π_I is distance nonincreasing, $\pi_I(\Sigma)$ is a 2-manifold in ∂R_I with area $O(L^2)$, so if we can create an annulus between $\partial\Sigma$ and $\pi_I(\partial\Sigma)$ which has area $O(L^3)$, then taking Σ' to be the union of this annulus with $\pi_I(\Sigma)$ completes the proof. By Lemma 11, there is a path from each point in $\partial\Sigma$ to its image in $\pi_I(\partial\Sigma)$ which has length $O(L)$. Two adjacent points in $\partial\Sigma$, along with their images in $\pi_I(\partial\Sigma)$ and these two paths give a loop of length $O(L)$ in $U_{\Phi(I)^+}\mathbf{A}_I(\mathcal{O}_S)B$ where B is a ball in $\mathbf{M}_I(\mathcal{O}_S)$ of radius r around 1. Note that this subset of G is quasi-isometric to $U_{\Phi(I)^+}\mathbf{A}_I(\mathcal{O}_S)$, and by Lemma 13, these loops have quadratic fillings in ∂R_I. Since there are $O(L)$ such loops formed by adjacent pairs of points in $\partial\Sigma$, this gives an annulus $\mathcal{A}$ with $\partial\mathcal{A} = \partial\Sigma \cup \pi_I(\partial\Sigma)$, and $area(\mathcal{A}) = O(L^3)$, completing the proof. $\square$

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